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q-deformed Minkowski space based on a q-Lorentz algebra

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Abstract. The Hilbert space representations of a non-commutative q-deformed Minkowski space, its momenta and its Lorentz boosts are constructed. The spectrum of the diagonalizable space elements shows a lattice-like structure with accumulation points on the light-cone.

1 Introduction

A non-commutative space-time structure emerges from quantum group considerations.

More precisely, if we demand that space-time variables are modules or co-modules of the q-deformed Lorentz group, then they satisfy commutation relations that make them elements of a non-comutative space. The action of momenta on this space is non-commuative as well. The full structure is determined by the (co-)module property.

This algebra has been constructed in [1]. It can serve as an explicit example of a non-commutative structure for space-time.

This has the advantages that the q-deformed Lorentz group plays the role of a cinematical group and thus determines many of the properties of this space and allows explicit calculations. We have explicitly constructed Hilbert space representations of the algebra and find that the vectors in the Hilbert space can be determined by measuring the time, the three-dimensional distance, the q-deformed angular momentum and its third component. The eigenvalues of these observables form a q-lattice with accumulation points on the light-cone. In a way physics on the light-cone is best approximated by this q-deformation. It is an interesting result that time-like and space-like regions serve as basis for irreducible representations independently. It will be shown however in a forthcoming paper [2] that these representations are linked together if we demand that the observables are essentially selfadjoint operators.

The paper is organized as follows. We first present the algebra. In Chap. 2 we give explicit formulas for the matrix elements of the elements of the algebra. This is the main result of this work and can serve as a starting point for further investigations. In the following chapters we give a rather detailed guide how these results can be obtained, first for the space-time algebra (Chap. 3), then for the Lorentz algebra (Chap. 4).

The algebra represented that far is isomorphic to the q-deformed Poincaré algebra. We would just have to replace X by P to obtain the respective representations [3].

In the next chapter (Chap. 5) we enlarge the algebra by a scaling operator and we introduce a canonical notation for labeling the states.

Finally in Chap. 6 we construct the representations of the momenta in the X-basis. There we learn that the full algebra cannot be represented on the light-cone by itself. The points on the light-cone are limiting points from the time-like and space-like regions.

2 The algebra

The algebra derived in [1] is generated by the elements X^a, (coordinates), P^a (momenta), V^{ab} (q-Lorentz transformations), and $\Lambda^{\frac{1}{2}}$ (scaling operator). [1]

The space is non-commutative:

$$\begin{aligned} \varepsilon_{CB}{}^{A} X^B X^C &= (1-q^2) X^0 X^A \\ X^0 X^C &= X^C X^0 \end{aligned} \tag{2.1}$$

The momenta are subject to the same relations:

$$\begin{aligned} \varepsilon_{CB}{}^{A} P^B P^C &= (1-q^2) P^0 P^A \\ P^0 P^C &= P^C P^0 \end{aligned} \tag{2.2}$$

The defining relations of the q-Lorentz algebra, as it acts on coordinates and momenta, are more easily expressed in the "Pauli"-notation:

$$\begin{aligned} V^{A0} &= R^A + q^2 S^A \\ V^{0A} &= -q^2 R^A - S^A \\ V^{AB} &= \varepsilon_C{}^{AB}(R^C - S^C) \\ V^{00} &= 0 \end{aligned} \tag{2.3}$$

[1] Capital letters ^{A} denote the three space indices $(+,-,3)$, small letters ^{a} denote the four Minkowski indices $(+,-,3,0)$. ε_{CBA} is the q-deformed ε-tensor and g_{AB} the Euclidean metric, η_{ab} the Lorentz metric. For the scalar product we write $X \circ Y = g_{AB} X^A Y^B$ (see also Appendix A)

In addition, we introduce an element U that is related to the Casimir operators of the q-Lorentz algebra:

$$U^2 = 1 + \frac{1}{2}(q^4-1)^2(R\circ R + S\circ S) \tag{2.4}$$

The q-Lorentz algebra[2]:

$$\begin{aligned} \varepsilon_{CB}{}^{A} R^B R^C &= \frac{1}{1+q^2} U R^A \\ \varepsilon_{CB}{}^{A} S^B S^C &= -\frac{1}{1+q^2} U S^A \\ R^A S^B &= q^2 \hat{R}^{AB}{}_{CD} S^C R^D \\ U R^A = R^A U, &\qquad U S^A = S^A U \end{aligned} \tag{2.5}$$

The coordinates "transform" under the q-Lorentz transformations:

$$\begin{aligned} R^A X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1} X^0 R^A + \frac{1}{q}\frac{q^2-1}{q^2+1}\varepsilon_{LM}{}^{A} X^M R^L \\ &\quad - \frac{q}{(1+q^2)^2} X^A U \\ R^A X^B &= \frac{1}{1+q^2}\Big[q(1+q^2)X^A R^B \\ &\quad - \frac{1}{q}(q^2-1)\varepsilon_C{}^{AB} X^0 R^C - \frac{1}{q}(q^2-1) g^{AB} X\circ R \\ &\quad - \frac{2}{q}\varepsilon^{ABG}\varepsilon_{STG} X^T R^S - \frac{1}{q}\frac{1}{1+q^2} g^{AB} X^0 U \\ &\quad + \frac{1}{q}\frac{1}{1+q^2}\varepsilon_M{}^{AB} X^M U\Big] \end{aligned}$$

$$\begin{aligned} S^A X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1} X^0 S^A + \frac{1}{q}\frac{q^2-1}{q^2+1}\varepsilon_{LM}{}^{A} X^M S^L \\ &\quad - \frac{1}{q(1+q^2)^2} X^A U \\ S^A X^B &= \frac{1}{1+q^2}\Big[\frac{1}{q}(1+q^2)X^A S^B \\ &\quad - \frac{1}{q}(q^2-1)\varepsilon_C{}^{AB} X^0 S^C + q(q^2-1) g^{AB} X\circ S \\ &\quad - \frac{2}{q}\varepsilon^{ABG}\varepsilon_{STG} X^T S^S - \frac{q}{1+q^2} g^{AB} X^0 U \\ &\quad - \frac{1}{q}\frac{1}{1+q^2}\varepsilon_M{}^{AB} X^M U\Big] \end{aligned} \tag{2.6}$$

$$\begin{aligned} U X^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1} X^0 U - \frac{1}{q}(q^2-1)^2 X\circ R \\ U X^A &= \frac{1}{q}\frac{q^4+1}{q^2+1} X^A U - q(q^2-1)^2 X^0 R^A \\ &\quad - \frac{1}{q}(q^2-1)^2 \varepsilon_{CB}{}^{A} X^B R^C \end{aligned}$$

The momenta have the same transformation law.

[2] The $\hat{R}$ matrices are also defined in Appendix A

The scaling operator acts as follows:

$$\begin{aligned} \Lambda^{\frac{1}{2}} X^a &= \frac{1}{q} X^a \Lambda^{\frac{1}{2}} \\ \Lambda^{\frac{1}{2}} P^a &= q P^a \Lambda^{\frac{1}{2}} \\ \Lambda^{\frac{1}{2}} V^{ab} &= V^{ab} \Lambda^{\frac{1}{2}} \\ \Lambda^{\frac{1}{2}} U &= U \Lambda^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

The relations that generalizes the Heisenberg commutation relations are: [2]

$$\begin{aligned} &P^a X^b - q^{-2}\hat{R}_{II}^{-1\,ab}{}_{cd} X^c P^d = \\ &-\tfrac{i}{2}\Lambda^{-\frac{1}{2}}\left\{(1+q^4)\eta^{ab}U + q^2(1-q^4)V^{ab}\right\} \end{aligned} \tag{2.8}$$

The q-Heisenberg algebra (2.8) does not separate from the q-Lorentz algebra for $q \neq 1$. The relation (2.8) tells us how to commute X^a and P^b and how to define orbital angular momentum in terms of the space and momentum operators. It is not possible to define V^{ab} in terms of an X, P ordered expression.

From orbital angular momentum we expect additional relations – the orbital angular momentum is orthogonal to the coordinates and momenta. These relations follows from the defining relations of our algebra and they are:

$$g_{AB} X^A (R^B - q^2 S^B) = 0 \tag{2.9}$$
$$X^0(S^A - q^2 R^A) - \varepsilon_{CB}{}^{A} X^B (R^C + S^C) = 0$$

The same with X replaced by P. Not all representations of the Lorentz group can be realized as angular momentum. We expect a relation for the Casimir operators. It follows from the algebra that:

$$R\circ R = S\circ S \tag{2.10}$$

For the physical interpretation and for the representations of this algebra the conjugation properties are very important. They are:

$$\begin{aligned} \overline{X^0} = X^0 \quad &, \quad \overline{X^A} = g_{AB} X^B \\ \overline{P^0} = P^0 \quad &, \quad \overline{P^A} = g_{AB} P^B \\ \overline{R^A} = -g_{AB} S^B \quad &, \quad \overline{S^A} = -g_{AB} R^B \\ \overline{U} &= U \\ \overline{\Lambda^{1/2}} &= q^4 \Lambda^{-1/2} \end{aligned} \tag{2.11}$$

These conjugation relations are consistent with the algebra.

Finally, we identify the three-dimensional rotations in the algebra. They have to commute with X^0 and $X\circ X$ Such operators have been found in [4] and they are:

$$L^A = \frac{q^2+1}{q^2}\left(U S^A - U R^A + (q^4-1)\varepsilon_{CB}{}^{A} R^B S^C\right) \tag{2.12}$$

They commute with X^0 and P^0 as well as with all "scalars" in our algebra formed with the metric g_{AB}, such as $X\circ X$, $P\circ P$, $R\circ R$, $S\circ R$, etc.

To write the L algebra in a familiar way we define an additional element:

$$W = U^2 - q^2(q^4-1)^2 R\circ S \tag{2.13}$$

and find:

$$\begin{aligned}\varepsilon_{BC}{}^{A} L^C L^B &= -\frac{W}{q^2} L^A \\ q^4(q^2-1)^2 L\circ L &= W^2 - 1\end{aligned} \tag{2.14}$$

The $SU_q(2)$ algebra was written in this form in [5,6]. We identify the $SU_q(2)$ generators:

$$\begin{aligned}T^+ &= q^2\sqrt{1+q^2} \quad \tau^{1/2} L^+ \\ T^- &= -q^3\sqrt{1+q^2} \quad \tau^{1/2} L^- \\ \tau_3 &= \tau\end{aligned} \tag{2.15}$$

with $\tau^{-\frac{1}{2}} = W + q^2(1-q^2)L^3$.

The T algebra is the familiar one:

$$\begin{aligned}\frac{1}{q}T^+T^- - qT^-T^+ &= \frac{1-\tau_3}{q-\frac{1}{q}} \\ \tau_3 T^+ &= \frac{1}{q^4}T^+\tau_3 \\ \tau_3 T^- &= q^4 T^- \tau_3\end{aligned} \tag{2.16}$$

Its Casimir operator is:

$$\mathbf{T}^2 = qT^-T^+ + \frac{q}{(q-\frac{1}{q})^2}\tau_3^{-\frac{1}{2}} + \frac{1}{q}\frac{1}{(q-\frac{1}{q})^2}\left(\tau_3^{\frac{1}{2}} - q^2 - 1\right) \tag{2.17}$$

The conjugation properties are:

$$\begin{aligned}\overline{T^+} &= \frac{1}{q^2}T^-, \quad \overline{T^-} = q^2 T^+ \\ \overline{\tau_3} &= \tau_3\end{aligned} \tag{2.18}$$

The vectors X^A and P^A transform as follows:

$$\begin{aligned}L^A X^B &= g^{AB} X\circ L - \frac{1}{q^2}\varepsilon_{KC}{}^{A}\varepsilon_{D}{}^{KB} X^C L^D \\ &\quad -\frac{1}{q^4}\varepsilon_{C}{}^{AB} X^C W \\ W X^A &= (q^2 + \frac{1}{q^2} - 1) X^A W + (q^2-1)^2 \varepsilon_{DC}{}^{A} X^C L^D \\ W X^0 &= X^0 W\end{aligned} \tag{2.19}$$

Now we have all the relations that allow us to study the representations of this algebra.

A complete set of commuting operators is X^0, $X\circ X$, $\mathbf{T}^2$ and τ_3.

3 The matrix elements

In this chapter we present the matrix elements of all the members of the algebra.

The states are labeled by the quantum numbers j, m, n and M. The quantum numbers j and m refer to the q-deformed angular momentum. The quantum numbers n and M label the eigenvalues of the time X^0 and the three-dimensional radius $X\circ X$.

There are inequivalent representations for the time-like and space-like regions.

Space-like: $s^2 = t^2 - r^2 < 0$:

$$\begin{aligned}M &= -\infty\ldots\infty \\ n &= -\infty\ldots\infty \\ j &= 0\ldots\infty\end{aligned}$$

$$\begin{aligned}X^0\ |j,m,n,M\rangle &= \frac{l_0 q^M}{[2]}\lambda[n]\ |j,m,n,M\rangle \\ X\circ X\ |j,m,n,M\rangle &= \frac{l_0^2 q^{2M}}{[2]^2}\{n+1\}\{n-1\}\ |j,m,n,M\rangle\end{aligned} \tag{3.1}$$

Time-like: $s^2 = t^2 - r^2 > 0$:

$$\begin{aligned}M &= -\infty\ldots\infty \\ n &= 0\ldots\infty \\ j &= 0\ldots n\end{aligned}$$

$$\begin{aligned}X^0\ |j,m,n,M\rangle &= \frac{t_0 q^M}{[2]}\{n+1\}\ |j,m,n,M\rangle \\ X\circ X\ |j,m,n,M\rangle &= \frac{t_0^2 q^{2M}\lambda^2}{[2]^2}[n+2][n]\ |j,m,n,M\rangle\end{aligned} \tag{3.2}$$

We use the notation throughout this paper:

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} \tag{3.3}$$

$$\{a\} = q^a + q^{-a} \tag{3.4}$$

and

$$\lambda = q - \frac{1}{q} \tag{3.5}$$

The spectrum of the operators X^0 and $X\circ X$ is shown in Fig. The parameters $|t_0|$, l_0 range from 1 to q and label inequivalent representations. t_0 can be positive (forward cone) and negative (backward cone).

The states are orthonormal:

$$\langle j',m',n',M'|j,m,n,M\rangle = \delta_{j',j}\delta_{m',m}\delta_{n',n}\delta_{M',M} \tag{3.6}$$

The matrix elements of X^A, R^A, S^A and P^A can be expressed in terms of reduced matrix elements. The explicit formulas are given in (4.5).

Space-like:

Reduced matrix elements of X^-:

$$\langle j+1,n,M\|X^-\|j,n,M\rangle =$$

$$l_0 q^{M+j}\frac{\sqrt{\{n+j+1\}\{n-j-1\}}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}} \qquad \text{for } j\geq 0$$

$$\langle j,n,M\|X^-\|j,n,M\rangle =$$

$$-q^{-1}\frac{l_0 q^M [n]\lambda^2}{\sqrt{[2]}\{j\}\{j+1\}} \qquad \text{for } j\geq 1$$

$$\langle j,n,M\|X^-\|j+1,n,M\rangle =$$

$$-l_0 q^{M-j-2}\frac{\sqrt{\{n+j+1\}\{n-j-1\}}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}} \qquad \text{for } j\geq 0$$

Reduced matrix elements of R^-:

$$\langle j+1,n',M\|R^-\|j,n,M\rangle =$$

$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{2j-1}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{(n'-n)(j+1)+n'\}\{(n'-n)j+n'\}}{\{n\}\{n'\}}}$$

for $j\geq 0$

$$\langle j,n',M\|R^-\|j,n,M\rangle =$$

$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}} \cdot\sqrt{\frac{\{(n'-n)j+n'\}\{n-(n'-n)j\}}{\{n\}\{n'\}}}$$

for $j\geq 1$

$$\langle j,n',M\|R^-\|j+1,n,M\rangle =$$

$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-2j-5}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{n'-(n'-n)(j+1)\}\{n-(n'-n)(j+1)\}}{\{n\}\{n'\}}}$$

for $j\geq 0$

Reduced matrix elements of S^-:

$$\langle j+1,n',M\|S^-\|j,n,M\rangle =$$

$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{(n'-n)(j+1)+n'\}\{(n'-n)j+n'\}}{\{n\}\{n'\}}}$$

for $j\geq 0$

$$\langle j,n',M\|S^-\|j,n,M\rangle =$$

$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}} \cdot\sqrt{\frac{\{(n'-n)j+n'\}\{n-(n'-n)j\}}{\{n\}\{n'\}}}$$

for $j\geq 1$

$$\langle j,n',M\|S^-\|j+1,n,M\rangle =$$

$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{n'-(n'-n)(j+1)\}\{n-(n'-n)(j+1)\}}{\{n\}\{n'\}}}$$

for $j\geq 0$

Reduced matrix elements of P^-:

$$\langle j+1,n+1,M'\|P^-\|j,n,M\rangle =$$

$$\frac{i}{2}(\delta_{M',M+1}q^{2+2j-n}-\delta_{M',M-1}q^{2+n}) \cdot\frac{1}{\{j+1\}\lambda l_0 q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{n+j+2\}\{n+j+1\}}{\{n\}\{n+1\}}}$$

for $j\geq 0$

$$\langle j+1,n-1,M'\|P^-\|j,n,M\rangle =$$

$$\frac{i}{2}(\delta_{M',M+1}q^{2+2j+n}-\delta_{M',M-1}q^{2-n}) \cdot\frac{1}{\{j+1\}\lambda l_0 q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}} \cdot\sqrt{\frac{\{n-j-2\}\{n-j-1\}}{\{n\}\{n-1\}}}$$

for $j\geq 0$

$$\langle j,n+1,M'\|P^-\|j,n,M\rangle =$$

$$\frac{i\lambda}{2\{j\}\{j+1\}\sqrt{[2]}l_0 q^M}(\delta_{M',M+1}q^{-n}+\delta_{M',M-1}q^{2+n}) \cdot\sqrt{\frac{\{n+j+1\}\{n-j\}}{\{n\}\{n+1\}}}$$

for $j\geq 1$

$$\langle j,n-1,M'\|P^-\|j,n,M\rangle =$$

$$-\frac{i\lambda}{2\{j\}\{j+1\}\sqrt{[2]}l_0q^M}(\delta_{M',M+1}q^n+\delta_{M',M-1}q^{2-n}) \cdot\sqrt{\frac{\{n-j-1\}\{n+j\}}{\{n\}\{n-1\}}}$$

for $j\geq 1$

$$\langle j,n+1,M'\|P^-\|j+1,n,M\rangle =$$

$$-\frac{i}{2}(\delta_{M',M+1}q^{-2-2j-n}-\delta_{M',M-1}q^{2+n}) \cdot\frac{1}{\{j+1\}\lambda l_0q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{\{n-j\}\{n-j-1\}}{\{n\}\{n+1\}}}$$

for $j\geq 0$

$$\langle j,n-1,M'\|P^-\|j+1,n,M\rangle =$$

$$-\frac{i}{2}(\delta_{M',M+1}q^{-2-2j+n}-\delta_{M',M-1}q^{2-n}) \cdot\frac{1}{\{j+1\}\lambda l_0q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{\{n+j+1\}\{n+j\}}{\{n\}\{n-1\}}}$$

for $j\geq 0$

Matrix elements of P^0:

$$\langle j,m,n+1,M'|P^0|j,m,n,M\rangle =$$

$$-\frac{i}{2\lambda l_0q^M}(\delta_{M',M+1}q^{1-n}+\delta_{M',M-1}q^{3+n}) \cdot\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}} \qquad \text{for } j\geq 0$$

$$\langle j,m,n-1,M'|P^0|j,m,n,M\rangle =$$

$$\frac{i}{2\lambda l_0q^M}(\delta_{M',M+1}q^{1+n}+\delta_{M',M-1}q^{3-n}) \cdot\sqrt{\frac{\{n-j-1\}\{n+j\}}{\{n\}\{n-1\}}} \qquad \text{for } j\geq 0$$

Matrix elements of U:

$$\langle j,m,n,M|U|j,m,n+1,M\rangle =$$

$$\langle j,m,n+1,M|U|j,m,n,M\rangle =$$

$$\frac{1}{[2]}\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}} \qquad \text{for } j\geq 0$$

Matrix elements of Λ:

$$\langle j,m,n,M+1|\Lambda^{\frac{1}{2}}|j,m,n,M\rangle = q^2$$

Time-like:

Reduced matrix elements of X^-:

$$\langle j+1,n,M\|X^-\|j,n,M\rangle =$$

$$t_0q^{M+j}\lambda\frac{\sqrt{[n-j][n+j+2]}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}}$$

for $j\geq 0$

$$\langle j,n,M\|X^-\|j,n,M\rangle =$$

$$-q^{-1}\lambda\frac{t_0q^M\{n+1\}}{\sqrt{[2]}\{j\}\{j+1\}} \qquad \text{for } j\geq 1$$

$$\langle j,n,M\|X^-\|j+1,n,M\rangle$$

$$= -t_0q^{M-j-2}\lambda\frac{\sqrt{[n-j][n+j+2]}}{\{j+1\}\sqrt{[2][2j+1][2j+3]}}$$

for $j\geq 0$

Reduced matrix elements of R^-:

$$\langle j+1,n',M\|R^-\|j,n,M\rangle =$$

$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{2j-1}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{[(n'-n)(j+1)+n'+1][(n'-n)j+n'+1]}{[n+1][n'+1]}}$$

for $j\geq 0$

$$\langle j,n',M\|R^-\|j,n,M\rangle =$$

$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}} \cdot\sqrt{\frac{[(n'-n)j+n'+1][n-(n'-n)j+1]}{[n+1][n'+1]}}$$

for $j\geq 1$

$$\langle j,n',M\|R^-\|j+1,n,M\rangle =$$

$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-2j-5}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}} \cdot\sqrt{\frac{[n'-(n'-n)(j+1)+1][n-(n'-n)(j+1)+1]}{[n+1][n'+1]}}$$

for $j\geq 0$

Reduced matrix elements of S^-:

$$\langle j+1,n',M\|S^-\|j,n,M\rangle =$$
$$(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}}$$
$$\cdot\sqrt{\frac{[(n'-n)(j+1)+n'+1][(n'-n)j+n'+1]}{[n+1][n'+1]}}$$
for $j\geq 0$

$$\langle j,n',M\|S^-\|j,n,M\rangle =$$
$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{q^{-3}}{\{j+1\}\{j\}[2]^{\frac{3}{2}}}$$
$$\cdot\sqrt{\frac{[(n'-n)j+n'+1][n-(n'-n)j+1]}{[n+1][n'+1]}}$$
for $j\geq 1$

$$\langle j,n',M\|S^-\|j+1,n,M\rangle =$$
$$-(\delta_{n',n+1}+\delta_{n',n-1})\frac{(n'-n)q^{-3}}{\{j+1\}[2]^{\frac{3}{2}}\lambda\sqrt{[2j+1][2j+3]}}$$
$$\cdot\sqrt{\frac{[n'-(n'-n)(j+1)+1][n-(n'-n)(j+1)+1]}{[n+1][n'+1]}}$$
for $j\geq 0$

Reduced matrix elements of P^-:

$$\langle j+1,n+1,M'\|P^-\|j,n,M\rangle =$$
$$\frac{i}{2}(\delta_{M',M+1}q^{1+2j-n}+\delta_{M',M-1}q^{3+n})$$
$$\cdot\frac{1}{t_0q^M\lambda\{j+1\}}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n+j+3][n+j+2]}{[n+2][n+1]}}$$
for $j\geq 0$

$$\langle j+1,n-1,M'\|P^-\|j,n,M\rangle =$$
$$-\frac{i}{2}(\delta_{M',M+1}q^{3+2j+n}+\delta_{M',M-1}q^{1-n})$$
$$\cdot\frac{1}{t_0q^M\lambda\{j+1\}}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n-j-1][n-j]}{[n][n+1]}}$$
for $j\geq 0$

$$\langle j,n+1,M'\|P^-\|j,n,M\rangle =$$
$$\frac{i\lambda}{2\{j\}\{j+1\}\sqrt{[2]}t_0q^M}(\delta_{M',M+1}q^{-1-n}-\delta_{M',M-1}q^{3+n})$$
$$\cdot\sqrt{\frac{[n+j+2][n-j+1]}{[n+2][n+1]}}\qquad\text{for } j\geq 1$$

$$\langle j,n-1,M'\|P^-\|j,n,M\rangle =$$
$$\frac{i\lambda}{2\{j\}\{j+1\}\sqrt{[2]}t_0q^M}(\delta_{M',M+1}q^{1+n}-\delta_{M',M-1}q^{1-n})$$
$$\cdot\sqrt{\frac{[n+j+1][n-j]}{[n][n+1]}}$$
for $j\geq 1$

$$\langle j,n+1,M'\|P^-\|j+1,n,M\rangle =$$
$$-\frac{i}{2}(\delta_{M',M+1}q^{-3-2j-n}+\delta_{M',M-1}q^{3+n})$$
$$\cdot\frac{1}{\lambda\{j+1\}t_0q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n-j+1][n-j]}{[n+2][n+1]}}$$
for $j\geq 0$

$$\langle j,n-1,M'\|P^-\|j+1,n,M\rangle =$$
$$\frac{i}{2}(\delta_{M',M+1}q^{-1-2j+n}+\delta_{M',M-1}q^{1-n})$$
$$\cdot\frac{1}{\lambda\{j+1\}t_0q^M}\sqrt{\frac{[2]}{[2j+1][2j+3]}}\sqrt{\frac{[n+j+2][n+j+1]}{[n][n+1]}}$$
for $j\geq 0$

Matrix elements of P^0:

$$\langle j,m,n+1,M'|P^0|j,m,n,M\rangle =$$
$$-\frac{i}{2\lambda t_0q^M}(\delta_{M',M+1}q^{-n}-\delta_{M',M-1}q^{4+n})$$
$$\cdot\sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}\qquad\text{for } j\geq 0$$

$$\langle j,m,n-1,M'|P^0|j,m,n,M\rangle =$$
$$-\frac{i}{2\lambda t_0q^M}(\delta_{M',M+1}q^{2+n}-\delta_{M',M-1}q^{2-n})$$
$$\cdot\sqrt{\frac{[n-j][n+j+1]}{[n][n+1]}}\qquad\text{for } j\geq 0$$

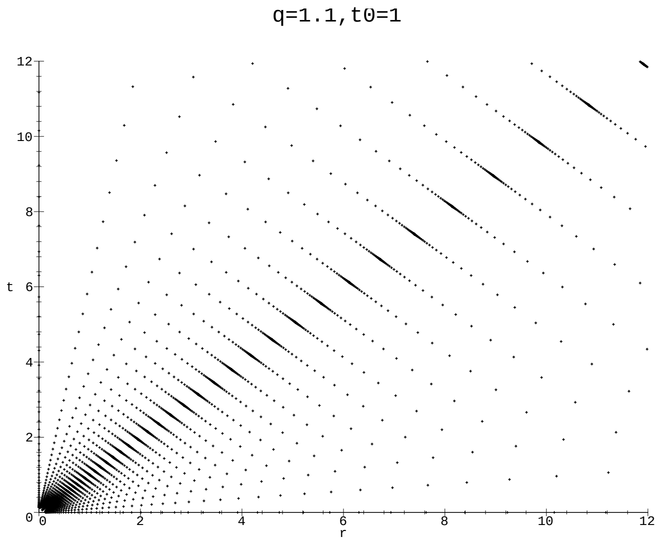

Fig. 1. Admissible values of t versus those of r for $q = 1.1$ and $t_0 = 1$

Matrix elements of U:

$$\langle j,m,n,M|U|j,m,n+1,M\rangle =$$

$$\langle j,m,n+1,M|U|j,m,n,M\rangle =$$

$$\frac{1}{[2]}\sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}$$

Matrix elements of Λ:

$$\langle j,m,n,M+1|\Lambda^{\frac{1}{2}}|j,m,n,M\rangle = q^2 \tag{3.7}$$

4 Matrix elements of the coordinates

In this chapter we indicate how to construct the matrix elements of the coordinates X.

We assume X^0 and $X \circ X$, as well as the elements $\mathbf{T}^2$ and τ ($\tau = \tau_3$) to be diagonal and label the states with the respective eigenvalues.

$$\begin{aligned}
\mathbf{T}^2|j,m,r,t\rangle &= [j][j+1]|j,m,r,t\rangle \\
\tau|j,m,r,t\rangle &= q^{-4m}|j,m,r,t\rangle \\
X^0|j,m,r,t\rangle &= t|j,m,r,t\rangle \\
X\circ X|j,m,r,t\rangle &= r^2|j,m,r,t\rangle
\end{aligned} \tag{4.1}$$

The well known representations of the T algebra are given in the appendix. As in the undeformed case, the T, X^A algebra allows us to express the "vector"-operator X^A through reduced matrix elements [7].

The respective algebra as it follows from Chap. 1 is:

$$\begin{aligned}
\tau X^3 &= X^3\tau \\
\tau X^+ &= q^{-4}X^+\tau \\
\tau X^- &= q^4X^-\tau
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
T^-X^3 &= X^3T^- + q\sqrt{1+q^2}X^- \\
T^+X^- &= q^2X^-T^+ + q^{-1}\sqrt{1+q^2}X^3 \\
T^-X^- &= q^2X^-T^-
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
T^+X^3 &= X^3T^+ + q^{-2}\sqrt{1+q^2}X^+ \\
T^+X^+ &= q^{-2}X^+T^+ \\
T^-X^+ &= q^{-2}X^+T^- + \sqrt{1+q^2}X^3
\end{aligned} \tag{4.4}$$

We proceed exactly as in the undeformed case. From (4.2) follows that X^3 does not change the eigenvalue of τ and that X^+, (X^-) changes m by $+1$, (-1).

From (4.4) we learn that the m dependence of the X^+ matrix elements can be computed explicitly and that the matrix elements of X^3 can be expressed in terms of the reduced matrix elements of X^+. From (4.3) follow the same relations for X^-. Via X^3, the reduced matrix elements of X^+ are related to the reduced matrix elements of X^-. As X^A commutes with $X \circ X$ and X^0, X^A does not change the eigenvalues of X^0 and $X \circ X$.

For the non-vanishing matrix elements we obtain the following result:

$$\begin{aligned}
&\langle j,m+1,r,t|X^+|j,m,r,t\rangle = \\
&-q^{m+2}\sqrt{[j+m+1][j-m]}\langle j,r,t\|X^-\|j,r,t\rangle \\
&\langle j+1,m+1,r,t|X^+|j,m,r,t\rangle = \\
&q^{m-2j}\sqrt{[j+m+1][j+m+2]}\langle j+1,r,t\|X^-\|j,r,t\rangle \\
&\langle j-1,m+1,r,t|X^+|j,m,r,t\rangle = \\
&q^{m+2j+2}\sqrt{[j-m][j-m-1]}\langle j-1,r,t\|X^-\|j,r,t\rangle
\end{aligned}$$

$$\begin{aligned}
&\langle j,m-1,r,t|X^-|j,m,r,t\rangle = \\
&q^m\sqrt{[j+m][j-m+1]}\langle j,r,t\|X^-\|j,r,t\rangle \\
&\langle j+1,m-1,r,t|X^-|j,m,r,t\rangle = \\
&q^m\sqrt{[j-m+1][j-m+2]}\langle j+1,r,t\|X^-\|j,r,t\rangle \\
&\langle j-1,m-1,r,t|X^-|j,m,r,t\rangle = \\
&q^m\sqrt{[j+m][j+m-1]}\langle j-1,r,t\|X^-\|j,r,t\rangle
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\langle j,m,r,t|X^3|j,m,r,t\rangle = \\
&q^{\frac{3}{2}}\frac{\sqrt{1+q^2}}{q^2-1}\{q^{2m}-\frac{q^{2j+1}+q^{-2j-1}}{q+q^{-1}}\}\langle j,r,t\|X^-\|j,r,t\rangle \\
&\langle j+1,m,r,t|X^3|j,m,r,t\rangle = \\
&q^{m-j-\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m+1][j+m+1]} \\
&\cdot\langle j+1,r,t\|X^-\|j,r,t\rangle \\
&\langle j-1,m,r,t|X^3|j,m,r,t\rangle = \\
&-q^{m+j+\frac{1}{2}}\sqrt{1+q^2}\sqrt{[j-m][j+m]}\langle j-1,r,t\|X^-\|j,r,t\rangle
\end{aligned}$$

All the m dependence of the X^A matrix elements is now explicitly known.

To get information on the reduced matrix elements we have to use the X, X relations (2.1).

We start with the relation:

$$X^3 X^+ - q^2 X^+ X^3 = (1-q^2) X^0 X^+ \qquad (4.6)$$

Depending on what matrix elements we take, (4.6) leads to a recursion formula for $\langle j,r,t\|X^-\|j,r,t\rangle$ or for the quantity $\rho_{r,t}$ which is defined as follows:

$$\begin{aligned} &\rho_{r,t}(j+1) \\ &\quad = [2j+1][2j+3] \qquad (4.7) \\ &\quad\quad \times \langle j,r,t\|X^-\|j+1,r,t\rangle\langle j+1,r,t\|X^-\|j,r,t\rangle \end{aligned}$$

These recursion formulas can be solved and we obtain:

$$\langle j,r,t\|X^-\|j,r,t\rangle = -\lambda q^{-1}\sqrt{[2]}\frac{[j][j+1]}{[2j][2j+2]}t \qquad (4.8)$$

As X^- changes the eigenvalue of τ the above matrix element has to be zero for $j=0$. Equation (4.8) is valid for $j \geq 1$.

$$\rho_{r,t}(j+1) = \rho_{r,t}(1) + \lambda^2[2]q^{-2}t^2\sum_{l=1}^{j}\frac{[2l+1]}{\{l\}^2\{l+1\}^2} \qquad (4.9)$$

The quantity $\rho_{r,t}(1)$ is the unknown left. It is related to the radius r. To see this we decompose $X \circ X$ into the product of matrix elements of X^A. The calculation is particularly simple for the matrix element:

$$\langle 0,0,r,t|X\circ X|0,0,r,t\rangle = r^2 = -q^2[2]\rho_{r,t}(1) \qquad (4.10)$$

For $j \neq 0$ the same calculation, but more tedious, yields:

$$\rho_{r,t}(j+1) = \frac{1}{q^2[2]}\left\{-r^2 + \frac{[j][j+2]\lambda^2}{\{j+1\}^2}t^2\right\} \qquad (4.11)$$

The two formulas (4.9) and (4.11) agree because the sum in (4.9) can be summed up:

$$\sum_{l=1}^{j}\frac{[2l+1]}{\{l\})^2\{l+1\}^2} = \frac{[2j]}{[2]\{j\}^2\{j+1\}^2}(1+\frac{[2j+2]}{[2]}) \qquad (4.12)$$

Equation (4.12) can be proved by induction.

From (4.10) follows that $\rho_{r,t}(1)$ is negative. We shall show that this is true for $\rho_{r,t}(j)$ for any value of j. It follows from the hermiticity properties of the coordinates:

$$\overline{X^3} = X^3, \qquad \overline{X^+} = -qX^-, \qquad \overline{X^-} = -\frac{1}{q}X^+ \qquad (4.13)$$

For the reduced matrix elements this implies:

$$\begin{aligned} \overline{\langle j,r,t\|X^-\|j,r',t'\rangle} &= \langle j,r',t'\|X^-\|j,r,t\rangle \qquad (4.14) \\ \overline{\langle j+1,r,t\|X^-\|j,r',t'\rangle} &= -q^{2(j+1)}\langle j,r',t'\|X^-\|j+1,r,t\rangle \end{aligned}$$

From the definition of $\rho_{r,t}$ (4.7) we now obtain:

$$\begin{aligned} &\rho_{r,t}(j+1) \qquad (4.15) \\ &\quad = -q^{-2(j+1)}[2j+1][2j+3]|\langle j+1,r,t\|X^-\|j,r,t\rangle|^2 \end{aligned}$$

Thus $\rho_{r,t}(j)$ is negative or zero. This can lead to an upper bound for j.

If we combine (4.15) with (4.11) we see that $|\langle j+1,r,t\|X^-\|j+1,r,t\rangle|^2$ is now explicitly known as a function of j, r and t. Only a phase is undetermined. But the relative phase between states with different j has not been fixed yet. We do it now, by assuming that $\langle j+1,r,t\|X^-\|j,r,t\rangle$ is real. Equation (4.14) then determines $\langle j,r,t\|X^-\|j+1,r,t\rangle$.

As the reduced matrix elements $\langle j,r,t\|X^-\|j,r,t\rangle$ have been given in (4.8), all the matrix elements of X^A are known as functions of j, m, r and t.

We have to learn more about the spectrum of t and r. This can be done by studying the X, R algebra.

We start with the following relations, they are a consequence of our algebra:

$$\begin{aligned} UX^0 &= \frac{1}{q}\frac{q^4+1}{q^2+1}X^0U - \frac{1}{q}(q^2-1)^2 X\circ R \qquad (4.16) \\ X\circ R\, X^0 &= \frac{2q}{1+q^2}X^0\ X\circ R - \frac{q}{(1+q^2)^2}X\circ X\ U \end{aligned}$$

If we take matrix elements of these relations we get two homogeneous linear equations in the matrix elements of U and $X\circ R$ that have a non trivial solution only if the determinant of the coefficient matrix vanishes:

$$\left(t - \frac{\lambda}{[2]}t'\right)\left(t - \frac{\{2\}}{[2]}t'\right) - \frac{\lambda^2}{[2]^2}{r'}^2 = 0 \qquad (4.17)$$

The invariant length commutes with $X\circ R$ and U, and, as a consequence

$$s^2 = t^2 - r^2 = {s'}^2 = {t'}^2 - {r'}^2 = -l^2 \qquad (4.18)$$

We shall use as a variable s^2 for the time-like and $l^2 = -s^2$ for the space-like case.

If we replace ${r'}^2$ in (4.17) by s^2 we obtain a quadratic equation in t that has the solution:

$$t = \frac{[2]}{2}t' \pm \frac{\lambda}{2}\sqrt{{t'}^2 - \left(\frac{2}{[2]}\right)^2 s^2} \qquad (4.19)$$

Thus t and t' have to be related this way for a non-vanishing matrix element $X\circ R$. For ${r'}^2 = 0$ however there is a special situation. From (4.11) follows that $\rho_{0,t}(1) = 0$ and that j has to be zero for $\rho_{0,t}$ not to be positive. From (4.15) all the X matrix elements are zero and thus the $X\circ R$ matrix elements as well:

$$\langle 0,0,0,t'|X\circ R|0,0,r,t\rangle = 0 \qquad (4.20)$$

The second equation of (4.16) is trivially satisfied leaving us with the equation:

$$\langle 0,0,0,t'|U|0,0,r,t\rangle\left(t - \frac{\{2\}}{[2]}t'\right) = 0 \qquad (4.21)$$

In this case the matrix element of U can only be different from zero if

$$t = \frac{\{2\}}{[2]} t' \tag{4.22}$$

In all the other cases, the matrix element of U is related to the matrix element of $X \circ R$:

$$\langle j, m, r', t' | U | j, m, r, t \rangle \left(t - \frac{\{2\}}{[2]} t' \right) = \tag{4.23}$$
$$-q\lambda^2 \langle j, m, r', t' | X \circ R | j, m, r, t \rangle$$

We shall now discuss the time-like, space-like and light-like region separately.

Let us start with the time-like region $s^2 > 0$. We assume that there is a point $r' = 0$ on the hyperbola, thus $t'_0 = s$ ($t'_0 = -s$).

According to the discussion above there is only the matrix element of U that connects this state to the state to the time:

$$t_1 = \frac{\{2\}}{[2]} s > s \tag{4.24}$$

We now continue to use (4.19) and find the other values of t:

$$t_n = \frac{s}{[2]} \{n+1\} \qquad n = 0, 1, \ldots, \infty \tag{4.25}$$

The values for r_n follow from (4.18):

$$r_n^2 = \frac{s^2 \lambda^2}{[2]^2} [n+2][n] \tag{4.26}$$

For the backward light-cone we just have to take t_0 negative.

If we would not have assumed $r' = 0$ to be in the spectrum our matrix elements would connect to negative values of r^2.

For the space-like region, $s^2 = -l^2 < 0$, a similar analysis gives the following values for t and r:

$$t_n = \pm \frac{l\lambda}{[2]} [n] \qquad n = -\infty \ldots \infty \tag{4.27}$$
$$r_n^2 = \frac{l^2}{[2]^2} \{n+1\}\{n-1\}$$

On the light cone, $s^2 = 0$:

$$t_n = q^n \tau_0 \qquad n = -\infty \ldots \infty \tag{4.28}$$
$$r_n^2 = q^{2n} \tau_0^2$$

If we now go back to (4.11) and insert the values of r_n and t_n we find that for the time-like region $\rho(n+1) = 0$. That means that in this case j is restricted to be $j \leq n$. There is no restriction of this type for the space-like region of the light-cone.

To conclude this section we give an explicit form for ρ for the time-like, space-like and light-like region. We find:

$$\begin{aligned}
&\text{space-like:} \quad \rho_n(j+1) = -\frac{l^2}{[2]q^2} \frac{\{n-j-1\}\{n+j+1\}}{\{j+1\}^2} \\
&\text{time-like:} \quad \rho_n(j+1) = -\frac{s^2\lambda}{[2]q^2} \frac{[n-j-1][n+j+2]}{\{j+1\}^2} \\
&\text{light-like:} \quad \rho_n(j+1) = -\frac{[2]\tau_0^2}{q^2} \frac{1}{\{j+1\}^2}
\end{aligned} \tag{4.29}$$

We see that only for the time-like region ρ_n can change sign.

From (4.15) follows with our phase convention:

$$\langle j+1, r, t \| X^- \| j, r, t \rangle = q^{j+1} \sqrt{\frac{-\rho_{r,t}(j+1)}{[2j+1][2j+3]}} \tag{4.30}$$
$$\langle j, r, t \| X^- \| j+1, r, t \rangle = -q^{-j-1} \sqrt{\frac{-\rho_{r,t}(j+1)}{[2j+1][2j+3]}}$$

We already know $\langle j, r, t \| X^- \| j, r, t \rangle$ (4.8). Then all the matrix elements of X depend on s for the time-like, on l for the space-like and on τ_0 for the light-like region as the only undetermined variable.

5 Matrix elements of the generators R^A of the q-Lorentz algebra

The operators R^A are "vector" operators as well, and their matrix elements can be expressed through reduced matrix elements. The formulas (4.5) are valid for R^A except that R^A is not diagonal in r and t.

If we analyze the "scalar" product of two arbitrary "vector" operators through matrix elements we get the general formula:

$$\begin{aligned}
&\langle j, m, \mu | A \circ B | j, m, \nu \rangle = \\
&\sum_{\nu'} \langle j, \mu \| A^- \| j, \nu' \rangle \langle j, \nu' \| B^- \| j, \nu \rangle \frac{1}{[2]} q^2 [2j+2][2j] \\
&- \langle j, \mu \| A^- \| j+1, \nu' \rangle \langle j+1, \nu' \| B^- \| j, \nu \rangle q^2 [2j+2][2j+3] \\
&- \langle j, \mu \| A^- \| j-1, \nu' \rangle \langle j-1, \nu' \| B^- \| j, \nu \rangle q^2 [2j][2j-1]
\end{aligned} \tag{5.1}$$

where ν, μ stand for the quantum numbers t and r.

We can apply (5.1) to $X \circ R$ and find:

$$\begin{aligned}
&\langle j, m, \mu | X \circ R | j, m, \nu \rangle = \\
&\langle j, \mu \| X^- \| j, \mu \rangle \langle j, \mu \| R^- \| j, \nu \rangle \frac{1}{[2]} q^2 [2j+2][2j] \\
&- \langle j, \mu \| X^- \| j+1, \mu \rangle \langle j+1, \mu \| R^- \| j, \nu \rangle q^2 [2j+2][2j+3] \\
&- \langle j, \mu \| X^- \| j-1, \mu \rangle \langle j-1, \mu \| R^- \| j, \nu \rangle q^2 [2j][2j-1]
\end{aligned} \tag{5.2}$$

This shows that in general ($r_\mu \neq 0$) non vanishing matrix elements of R^A will lead to non vanishing matrix elements of $X \circ R$. We know from the last section that $X \circ R$ has only non vanishing matrix elements between states labeled by

t_n and $t_{n\pm 1}$. Thus the non vanishing matrix elements for R^A are between these states as well.

We now use the R,X relations to get information on the R matrix elements. First the algebraic relation:

$$R^+X^+ = qX^+R^+ \tag{5.3}$$

If we take the $(j+2,j)$ matrix elements of this equation we obtain the recursion formulas:

$$\frac{\langle j+2,r',t'\|R^-\|j+1,r,t\rangle}{\langle j+1,r',t'\|R^-\|j,r,t\rangle} = q\frac{\langle j+2,r',t'\|X^-\|j+1,r',t'\rangle}{\langle j+1,r,t\|X^-\|j,r,t\rangle} \tag{5.4}$$

and:

$$\frac{\langle j-2,r',t',\|R^-\|j-1,r,t\rangle}{\langle j-1,r',t'\|R^-\|j,r,t\rangle} = q\frac{\langle j-2,r',t'\|X^-\|j-1,r',t'\rangle}{\langle j-1,r,t\|X^-\|j,r,t\rangle} \tag{5.5}$$

These formulas can be iterated. With the matrix elements of X^- expressed in terms of ρ, we find for $j>1$:

$$\frac{\langle j+1,r',t'\|R^-\|j,r,t\rangle}{\langle 1,r',t'\|R^-\|0,r,t\rangle} = q^{2j}\sqrt{\frac{[3]}{[2j+1][2j+3]}}\sqrt{\frac{\rho_{r',t'}(j+1)\dots\rho_{r',t'}(2)}{\rho_{r,t}(j)\dots\rho_{r,t}(1)}} \tag{5.6}$$

and

$$\frac{\langle j,r',t'\|R^-\|j+1,r,t\rangle}{\langle 0,r',t'\|R^-\|1,r,t\rangle} = q^{-2j}\sqrt{\frac{[3]}{[2j+1][2j+3]}}\sqrt{\frac{\rho_{r,t}(j+1)\dots\rho_{r,t}(2)}{\rho_{r',t'}(j)\dots\rho_{r',t'}(1)}}$$

There is another relation that follows from (5.3) if we take the $(j+1,j)$ matrix elements. It is:

$$\begin{aligned}&\langle j+1,r',t'\|R^-\|j+1,r,t\rangle\langle j+1,r,t\|X^-\|j,r,t\rangle\\ &+\langle j+1,r',t'\|R^-\|j,r,t\rangle\langle j,r,t\|X^-\|j,r,t\rangle = \\ &q\langle j+1,r',t'\|X^-\|j+1,r',t'\rangle\langle j+1,r',t'\|R^-\|j,r,t\rangle\\ &+q\langle j+1,r',t'\|X^-\|j,r',t'\rangle\langle j,r',t'\|R^-\|j,r,t\rangle\end{aligned} \tag{5.7}$$

This equation is valid for $j\geq 1$ and relates $(j+1,j+1)$,(j,j) and $(j+1,j)$ matrix elements of R^-.

If we study the relation:

$$R^+(X^3-X^0) = \frac{1}{q}(X^3-X^0)R^+ \tag{5.8}$$

and its $(j+1,j)$ matrix elements, the same R^- matrix elements as (5.7) are related. They can be combined to eliminate the $(j+1,j+1)$ matrix elements and to give a relation between the (j,j) and $(j+1,j)$ matrix elements of R^-:

$$\begin{aligned}&\langle j,r',t'\|R^-\|j,r,t\rangle\langle j+1,r',t'\|X^-\|j,r',t'\rangle q^{j+2}\\ &= \langle j+1,r',t'\|R^-\|j,r,t\rangle\sqrt{[2]}\left(\frac{t'\{j\}-t\{j+1\}}{\{j\}\{j+1\}}\right)\end{aligned} \tag{5.9}$$

It is valid for $j\geq 1$.

Taking the corresponding $(j-1,j)$ matrix elements we obtain:

$$\begin{aligned}&\langle j,r',t'\|R^-\|j,r,t\rangle\langle j-1,r',t'\|X^-\|j,r',t'\rangle\\ &= \langle j-1,r',t'\|R^-\|j,r,t\rangle\sqrt{[2]}\left(\frac{t'\{j+1\}-t\{j\}}{\{j\}\{j+1\}}\right)q^{j-1}\end{aligned} \tag{5.10}$$

This equation is valid for $j>1$.

Both equations can be used to find the (j,j) matrix elements from (5.6) in terms of the $(1,0)$ or $(0,1)$ matrix elements of R^-.

$$\begin{aligned}&\langle j,r',t'\|R^-\|j,r,t\rangle\\ &= \langle 1,r',t'\|R^-\|0,r,t\rangle\frac{1}{r}\frac{t'\{j\}-t\{j+1\}}{\{j\}\{j+1\}}\frac{[2]\sqrt{[3]}}{q^2}\\ &\cdot\sqrt{\frac{\rho_{r',t'}(j)\dots\rho_{r',t'}(2)}{\rho_{r,t}(j)\dots\rho_{r,t}(2)}}\\ &= -\langle 0,r',t'\|R^-\|1,r,t\rangle\frac{1}{r'}\frac{t'\{j+1\}-t\{j\}}{\{j\}\{j+1\}}q^2[2]\sqrt{[3]}\\ &\cdot\sqrt{\frac{\rho_{r,t}(j)\dots\rho_{r,t}(2)}{\rho_{r',t'}(j)\dots\rho_{r',t'}(2)}}\end{aligned} \tag{5.11}$$

For the values of ρ given in (4.11) it can be seen by induction in j that the relation between $\langle 1,r',t'\|R^-\|0,r,t\rangle$ and $\langle 0,r',t'\|R^-\|1,r,t\rangle$ that follows from (5.11) is indeed independent of j. We take $j=2$ and obtain:

$$\begin{aligned}&\langle 1,r',t'\|R^-\|0,r,t\rangle = \\ &\langle 0,r',t'\|R^-\|1,r,t\rangle(-q^4)\frac{r}{r'}\frac{t'\{3\}-t\{2\}}{t'\{2\}-t\{3\}}\frac{\rho_{r,t}(2)}{\rho_{r',t'}(2)}\end{aligned} \tag{5.12}$$

One of the matrix elements, e.g. $\langle 1,r',t'\|R^-\|0,r,t\rangle$, remains to be determined.

We already know that the U matrix elements are related to the R^- matrix elements from (4.23) and that U is hermitean.

$$\langle 0,0,t,r|U|0,0,t',r'\rangle = \overline{\langle 0,0,t',r'|U|0,0,t,r\rangle} \tag{5.13}$$

If we now use the relation:

$$UR^+ = R^+U \tag{5.14}$$

we find:

$$\Gamma(n) = |\langle 0,0,t_n,r_n|U|0,0,t_{n+1},r_{n+1}\rangle|^2 = \Gamma(n+1) \tag{5.15}$$

Thus $\Gamma(n)$ is n independent. This is valid for the time-like, space-like and light-like regions.

To finally determine $\Gamma(n)$ we have to use a relation that fixes the length of $R\circ R$. This relation is:

$$U^2-1 = (q^4-1)^2R\circ R \tag{5.16}$$

This is now sufficient to determine $\Gamma(n)$. We find:

$$\Gamma(n) = \frac{1}{[2]^2} \tag{5.17}$$

We can use the freedom of choosing the phase of states with different n eigenvalues such that:

$$\langle 0,0,r,t|U|0,0,r',t'\rangle = \frac{1}{[2]} \tag{5.18}$$

This determines all the matrix elements of U, R^A and S^A, as the S^A matrix elements are conjugate to the R^A matrix elements (2.11).

We finally give the explicit form of the following R^- matrix elements:

For $s^2 = 0$

$$\begin{aligned}
\langle 1,r_n,t_n\|R^-\|0,r_{n-1},t_{n-1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}\lambda}\\
\langle 1,r_n,t_n\|R^-\|0,r_{n+1},t_{n+1}\rangle &= -\tfrac{1}{q^2}\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}\lambda}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n-1},t_{n-1}\rangle &= -\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^6\lambda}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n+1},t_{n+1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^4\lambda}
\end{aligned} \tag{5.19}$$

For s^2 time-like

$$\begin{aligned}
\langle 1,r_n,t_n\|R^-\|0,r_{n-1},t_{n-1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q\lambda}\sqrt{\tfrac{[n+2]}{[n]}}\\
\langle 1,r_n,t_n\|R^-\|0,r_{n+1},t_{n+1}\rangle &= -\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q\lambda}\sqrt{\tfrac{[n]}{[n+2]}}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n-1},t_{n-1}\rangle &= -\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^5\lambda}\sqrt{\tfrac{[n-1]}{[n+1]}}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n+1},t_{n+1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^5\lambda}\sqrt{\tfrac{[n+3]}{[n+1]}}
\end{aligned} \tag{5.20}$$

For s^2 space-like

$$\begin{aligned}
\langle 1,r_n,t_n\|R^-\|0,r_{n-1},t_{n-1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q\lambda}\sqrt{\tfrac{\{n+1\}}{\{n-1\}}}\\
\langle 1,r_n,t_n\|R^-\|0,r_{n+1},t_{n+1}\rangle &= -\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q\lambda}\sqrt{\tfrac{\{n-1\}}{\{n+1\}}}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n-1},t_{n-1}\rangle &= -\tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^5\lambda}\sqrt{\tfrac{\{n-2\}}{\{n\}}}\\
\langle 0,r_n,t_n\|R^-\|1,r_{n+1},t_{n+1}\rangle &= \tfrac{1}{[2]^{\frac{5}{2}}\sqrt{[3]}q^5\lambda}\sqrt{\tfrac{\{n+2\}}{\{n\}}}
\end{aligned} \tag{5.21}$$

We could have started from the momenta instead of the coordinates, then we would have constructed representations of the q-deformed Poincaré algebra. Such representations are obtained by replacing X^a everywhere with P^a [8].

It should be noted that the representations with positive mass square ${p^0}^2 - \mathbf{p}^2 > 0$, have angular momentum limited by $j \le n$ (see discussion after the (4.23)).

6 The scaling operator Λ and the spectrum of X^0, $X \circ X$

The action of the scaling operator $\Lambda^{\frac{1}{2}}$ on the states $|j,m,r,t\rangle$ is easily found from (2.7):

$$\Lambda^{\frac{1}{2}}|j,m,r,t\rangle = \alpha_{j,m,r,t}|j,m,qr,qt\rangle \tag{6.1}$$

From (2.11) and (6.1) follows:

$$|\alpha_{j,m,r,t}|^2 = q^4 \tag{6.2}$$

It is obvious that $\Lambda^{\frac{1}{2}}$ changes the value of s^2 by a factor q^2. This shows that the values of s and l in (4.25), (4.26) and (4.27) have to take the following values:

$$\begin{aligned} s &= t_0 q^M, \qquad M = -\infty \ldots \infty\\ l &= l_0 q^M \end{aligned} \tag{6.3}$$

It is only the light-cone that is left invariant under the action of $\Lambda^{\frac{1}{2}}$

The states can be labeled with j, m, n and M for $s^2 > 0$ and for $s^2 < 0$. For $s^2 = 0$, j, m and n are sufficient.

For $s^2 > 0$:

$$\begin{aligned} M &= -\infty \ldots \infty\\ n &= 0 \ldots \infty\\ j &= 0 \ldots n \end{aligned}$$

$$\begin{aligned}
X^0\ |j,m,n,M\rangle &= \frac{t_0 q^M}{[2]}\{n+1\}\ |j,m,n,M\rangle\\
X \circ X\ |j,m,n,M\rangle &= \frac{t_0^2 q^{2M}\lambda^2}{[2]^2}[n+2][n]\ |j,m,n,M\rangle\\
\Lambda^{\frac{1}{2}}\ |j,m,n,M\rangle &= q^2|j,m,n,M+1\rangle
\end{aligned} \tag{6.4}$$

For $s^2 < 0$:

$$\begin{aligned} M &= -\infty \ldots \infty\\ n &= -\infty \ldots \infty\\ j &= 0 \ldots \infty \end{aligned}$$

$$\begin{aligned}
X^0\ |j,m,n,M\rangle &= \frac{l_0 q^M}{[2]}\lambda[n]\ |j,m,n,M\rangle\\
X \circ X\ |j,m,n,M\rangle &= \frac{l_0^2 q^{2M}}{[2]^2}\{n+1\}\{n-1\}\ |j,m,n,M\rangle\\
\Lambda^{\frac{1}{2}}\ |j,m,n,M\rangle &= q^2|j,m,m,n,M+1\rangle
\end{aligned} \tag{6.5}$$

For $s^2 = 0$:

$$\begin{aligned} n &= -\infty \ldots \infty\\ j &= 0 \ldots \infty \end{aligned}$$

$$\begin{aligned}
X^0\ |j,m,n\rangle &= \tau_0 q^n\ |j,m,n\rangle\\
X \circ X\ |j,m,n\rangle &= \tau_0^2 q^{2n}\ |j,m,n\rangle\\
\Lambda^{\frac{1}{2}}\ |j,m,n\rangle &= e^{i\alpha_n} q^2|j,m,m,n+1\rangle
\end{aligned} \tag{6.6}$$

In this case we cannot use the freedom of phase for the states to have $\alpha = 0$.

As we shall need the U-matrix elements in the next section we list them here explicitly.

For $s^2 < 0$:

$$\begin{aligned}&\langle j,m,n,M|U|j,m,n+1,M\rangle = \\ &\langle j,m,n+1,M|U|j,m,n,M\rangle = \\ &\frac{1}{[2]}\sqrt{\frac{\{n-j\}\{n+j+1\}}{\{n\}\{n+1\}}}\end{aligned} \tag{6.7}$$

For $s^2 > 0$:

$$\begin{aligned}&\langle j,m,n,M|U|j,m,n+1,M\rangle = \\ &\langle j,m,n+1,M|U|j,m,n,M\rangle = \\ &\frac{1}{[2]}\sqrt{\frac{[n-j+1][n+j+2]}{[n+1][n+2]}}\end{aligned} \tag{6.8}$$

We see that for the time-like region the matrix element of U is zero for $n = j-1$. This is in agreement with the condition $n \geq j$.

For $s^2 = 0$:

$$\begin{aligned}&\langle j,m,n|U|j,m,n+1\rangle \\ &= \langle j,m,n+1,M|U|j,m,n,M\rangle = \frac{1}{[2]}\end{aligned} \tag{6.9}$$

We shall see that for $s^2 \neq 0$ these states are sufficient to construct a representation of the full algebra introduced in Chap. 1. For $s^2 = 0$ there is no representation of this algebra.

7 Matrix elements of the momenta

We first write the q-deformed Heisenberg relations (2.8) in a more explicit version:

$$\begin{aligned}&q^2[2]P^0X^0 - q\{2\}X^0P^0 + \lambda X\circ P \\ &= \frac{i}{2}[2]\{2\}q^4\Lambda^{-\frac{1}{2}}U\end{aligned} \tag{7.1}$$

$$\begin{aligned}&q^2[2]P^0X^A - q\{2\}X^AP^0 - \lambda q^2X^0P^A - \lambda\epsilon_{DC}{}^AX^CP^D \\ &= -\frac{i}{2}[2]^2q^6\lambda\Lambda^{-\frac{1}{2}}(q^2R^A + S^A)\end{aligned} \tag{7.2}$$

$$\begin{aligned}&q^2[2]P^AX^0 - q\{2\}X^0P^A - \lambda q^2X^AP^0 - \lambda\epsilon_{DC}{}^AX^CP^D \\ &= \frac{i}{2}[2]^2q^6\lambda\Lambda^{-\frac{1}{2}}(R^A + q^2S^A)\end{aligned} \tag{7.3}$$

$$\begin{aligned}&[2](P^AX^B - X^AP^B) + \frac{2}{q^3}\epsilon_{DC}{}^E\epsilon_E{}^{AB}X^CP^D \\ &+\frac{\lambda}{q^2}\left(g^{AB}X\circ P - g^{AB}X^0P^0 + \epsilon_C{}^{AB}(X^CP^0 + X^0P^C)\right) \\ &= -\frac{i}{2}[2]q^2\Lambda^{-\frac{1}{2}} \\ &\times\left(\{2\}g^{AB}U - q^2\lambda[2]\epsilon_C{}^{AB}(R^C - S^C)\right)\end{aligned} \tag{7.4}$$

All these relations contain P^0, in the relations (7.3) and (7.4) P^0 is multiplied by λ.

We rearrange these relations to obtain two relations containing P^0 and $X\circ P$ as the only unknowns.

First we contract (7.4) with g_{AB}:

$$\begin{aligned}&P\circ X - \tfrac{1}{q^3}\tfrac{\{2\}}{[2]}X\circ P - q^4\lambda\tfrac{[3]}{[2]}X^0P^0 \\ &= -\tfrac{i}{2}q^2[2][3]\Lambda^{-\frac{1}{2}}U\end{aligned} \tag{7.5}$$

Equation (7.1) and (7.5) together with their conjugates yield three independent equations:

$$P^0X^0 - X^0P^0 = \frac{i}{2}(q^4\Lambda^{-\frac{1}{2}} + \Lambda^{\frac{1}{2}})U \tag{7.6}$$

$$P\circ X - X\circ P = -\frac{i}{2}[3](q^4\Lambda^{-\frac{1}{2}} + \Lambda^{\frac{1}{2}})U \tag{7.7}$$

$$\lambda(X\circ P - X^0P^0) = \frac{i}{2}q^2[2](\Lambda^{\frac{1}{2}} - \Lambda^{-\frac{1}{2}})U \tag{7.8}$$

Equation (7.7) can be used to express $P\circ X$ in terms of $X\circ P$. Equation (7.8) is one of the wanted equations, the second one is obtained by multiplying (7.3) by X^Bg_{BA}:

$$\begin{aligned}&X\circ PX^0 - \frac{2}{q[2]}X^0X\circ P - \frac{\lambda}{[2]}X\circ XP^0 \\ &= iq^4\lambda[2]X\circ R\Lambda^{-\frac{1}{2}}\end{aligned} \tag{7.9}$$

This provides us with a system of two linear equations for the two unknowns, the matrix elements of P^0 and $X\circ P$. The determinant of this system of linear equations is proportional to $[2]t'(t-t') + \lambda s^2$. For $s^2 \neq 0$ the equations can be solved. For $s^2 = 0$ and $t = t'$ the determinant vanishes. The homogeneous part of the two equations becomes linear dependent. For the inhomogeneous part this would imply $\langle j,m,n|U|j,m,n+1\rangle = 0$, in clear contradiction to (6.9). We conclude that the $s^2 = 0$ representation of the X^a, R^A, S^A, U, Λ algebra cannot be extended to a representation of the full algebra. For $s^2 \neq 0$ we can calculate the matrix elements. They are consistent with (7.6) and the other algebra relations.

From the $X\circ P$ matrix elements we obtain the reduced matrix elements of P^-, hermiticity of P has to be used. This way we obtain all the matrix elements of P^A. Representations of the full algebra have now been constructed. Their explicit form is given in Chap. 2. It is interesting that the forward, backward time-like and the space-like regions provide inequivalent, irreducible representations by themselves.

Appendix A R-matrices, metric and ε-tensor

Euclidean space

For the Euclidean space the metric tensor is defined as:

$$g_{AB}: \quad g_{+-} = -q, \quad g_{33} = 1, \quad g_{-+} = -\frac{1}{q} \tag{A.1}$$

$$g^{AB}: \quad g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -\frac{1}{q}$$

$$g_{AB}g^{BC} = \delta_A^C = g^{CB}g_{BA}$$

With the metric indices can be raised and lowered:

$$X_A = g_{AB}X^B, X^A = g^{AB}X_B$$

and an invariant scalar product can be given:

$$X \circ Y = g_{AB}X^AY^B = X^3Y^3 - qX^+Y^- - \frac{1}{q}X^-Y^+ \quad \text{(A.2)}$$

The ε-tensor is defined as:

$$\begin{aligned} &\varepsilon_{+-}{}^3 = q, \quad \varepsilon_{-+}{}^3 = -q, \quad \varepsilon_{33}{}^3 = 1 - q^2, \\ &\varepsilon_{+3}{}^+ = 1, \quad \varepsilon_{3+}{}^+ = -q^2, \\ &\varepsilon_{-3}{}^- = -q^2, \quad \varepsilon_{3-}{}^- = 1. \end{aligned} \quad \text{(A.3)}$$

Indices of the ε-tensor can also be raised and lowered through the metric, e.g.:

$$\varepsilon_{ABC} = g_{CD}\varepsilon_{AB}{}^D$$

In terms of the metric and of the ε-tensor the three-dimensional $\hat{R}$-matrix of the q-Euclidean space can be written in the form:

$$\begin{aligned} \hat{R}^{AB}{}_{CD} = &\delta^A_C\delta^B_D \\ &-q^{-4}\varepsilon^{FAB}\varepsilon_{FDC} - q^{-4}(q^2-1)g^{AB}g_{CD} \end{aligned} \quad \text{(A.4)}$$

Minkowski space

For the q-deformed Minkowski space it turns out that two different R-matrices exist. Their projector decomposition is given by:

$$\begin{aligned} \hat{R}_I &= P_S + P_T - q^2P_+ - q^{-2}P_- \quad \text{(A.5)} \\ &= \mathbb{1} - (1+q^2)P_+ - (1 + \frac{1}{q^2})P_- \\ \hat{R}_{II} &= q^{-2}P_S + q^2P_T - P_+ - P_- \quad \text{(A.6)} \\ &= \frac{1}{q^2}\mathbb{1} + (q^2 - \frac{1}{q^2})P_T - (1 + \frac{1}{q^2})P_A \end{aligned}$$

In these definitions P_S,P_T,P_+,P_- are the projectors on the symmetric, trace, selfdual, antiselfdual eigenspaces respectively. This decomposition shows clearly that $\hat{R}_I$ cannot distinguish the symmetric while $\hat{R}_{II}$ cannot distinguish the antisymmetric eigenspaces, because they have the same eigenvalue, so that both matrices are necessary to distinguish all the spaces. The explicit expression of the projectors follows.

P_+:

$$\begin{array}{c|cccc} & 00 & C0 & 0D & CD \\ \hline 00 & 0 & 0 & 0 & 0 \\ A0 & 0 & \frac{q^2}{(1+q^2)^2}\delta^A_C & -\frac{1}{(1+q^2)^2}\delta^A_D & \frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^A \\ 0B & 0 & -\frac{q^4}{(1+q^2)^2}\delta^B_C & \frac{q^2}{(1+q^2)^2}\delta^B_D & -\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^B \\ AB & 0 & \frac{q^2\varepsilon_C{}^{AB}}{(1+q^2)^2} & -\frac{\varepsilon_D{}^{AB}}{(1+q^2)^2} & \frac{\varepsilon_{DC}{}^E\varepsilon_E{}^{AB}}{(1+q^2)^2} \end{array} \quad \text{(A.7)}$$

P_-:

$$\begin{array}{c|cccc} & 00 & C0 & 0D & CD \\ \hline 00 & 0 & 0 & 0 & 0 \\ A0 & 0 & \frac{q^2}{(1+q^2)^2}\delta^A_C & -\frac{q^4}{(1+q^2)^2}\delta^A_D & -\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^A \\ 0B & 0 & -\frac{1}{(1+q^2)^2}\delta^B_C & \frac{q^2}{(1+q^2)^2}\delta^B_D & \frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^B \\ AB & 0 & -\frac{\varepsilon_C{}^{AB}}{(1+q^2)^2} & \frac{q^2\varepsilon_D{}^{AB}}{(1+q^2)^2} & \frac{\varepsilon_{DC}{}^E\varepsilon_E{}^{AB}}{(1+q^2)^2} \end{array} \quad \text{(A.8)}$$

P_T:

$$\begin{array}{c|cccc} & 00 & C0 & 0D & CD \\ \hline 00 & \frac{q^2}{(1+q^2)^2} & 0 & 0 & -\frac{q^2}{(1+q^2)^2}g_{CD} \\ A0 & 0 & 0 & 0 & 0 \\ 0B & 0 & 0 & 0 & 0 \\ AB & -\frac{q^2}{(1+q^2)^2}g^{AB} & 0 & 0 & \frac{q^2}{(1+q^2)^2}g^{AB}g_{CD} \end{array} \quad \text{(A.9)}$$

It holds:

$$\mathbb{1} = P_S + P_T + P_+ + P_- \quad \text{(A.10)}$$

Using P_T it is possible to construct a 4-dimensional metric:

$$\begin{aligned} &\eta_{00} = -1, \quad \eta_{33} = 1 \\ &\eta_{+-} = -q, \quad \eta_{-+} = -\tfrac{1}{q} \\ &\eta^{ab} = \eta_{ab} \end{aligned} \quad \text{(A.11)}$$

which enables to raise and lower indices:

$$X_A = \eta_{AB}X^B, X^A = \eta^{AB}X_B \quad \text{(A.12)}$$

and to define an invariant scalar product in 4 dimensions:

$$\begin{aligned} X \cdot Y &= X^0Y^0 - X^3Y^3 + qX^+Y^- + \frac{1}{q}X^-Y^+ \quad \text{(A.13)} \\ &= -\eta_{ab}X^aY^b \end{aligned}$$

The sum of the selfdual and antiselfdual projectors defines the q-deformed antisymmetrizer:

$$P_A = P_+ + P_- \quad \text{(A.14)}$$

while their difference defines the q-deformed 4-dimensional ε-tensor:

$$\varepsilon^{ab}{}_{cd} = P^{ab}_{+\ cd} - P^{ab}_{-\ cd} \quad \text{(A.15)}$$

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